

# TECHNICAL RESEARCH REPORT

G-Snakes:

Nonholonomic Kinematic Chains on Lie Groups

*by P.S. Krishnaprasad and D.P. Tsakiris*

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# $G$ -Snakes: Nonholonomic Kinematic Chains on Lie Groups

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## Abstract

We consider kinematic chains evolving on a finite-dimensional Lie group  $G$  under nonholonomic constraints, where snake-like global motion is induced by *shape* variations of the system. In particular, we consider the case when the evolution of the system is restricted to a subspace  $h$  of the corresponding Lie algebra  $\mathcal{G}$ , where  $h$  is not a subalgebra of  $\mathcal{G}$  and it can generate the whole algebra under Lie bracketing. Such systems are referred to as  $G$ -snakes. Away from certain singular configurations of the system, the constraints specify a (partial) connection on a principal fiber bundle, which in turn gives rise to a geometric phase under periodic shape variations. This geometric structure can be exploited in order to solve the nonholonomic motion planning problem for such systems.

$G$ -snakes generalize the concept of nonholonomic Variable Geometry Truss assemblies, which are kinematic chains evolving on the Special Euclidean group  $SE(2)$  under nonholonomic constraints imposed by idler wheels. We examine in detail the cases of 3-dimensional groups with real non-abelian Lie algebras such as the Heisenberg group  $H(3)$ , the Special Orthogonal group  $SO(3)$  and the Special Linear group  $SL(2)$ .

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## 1 Introduction

Of significant interest among mechanical systems subject to nonholonomic constraints are those wherein variations of shape induce, under the influence of the constraints, a global motion of the system. A well-known example is that of a free-floating multibody system in space (e.g. robotic manipulators mounted on orbiting satellites), where periodic movements of the joints induce a reorientation of the system, under the nonholonomic constraint of conservation of angular momentum (Krishnaprasad [1990]; Marsden, Montgomery & Ratiu [1990]).

Inspired by the experimental work of Joel Burdick and his students at Caltech (Chirikjian & Burdick [1991]; Chirikjian & Burdick [1993]), a novel system that uses the above principle for land locomotion was introduced in (Krishnaprasad & Tsakiris [1994]). There, a Variable Geometry Truss (VGT) assembly consisting of longitudinal repetition of truss modules, each one of which is equipped with idler wheels and linear actuators in a planar parallel manipulator configuration, uses periodic changes of the shape of each module to produce global motion. The locomotion principle is not based on direct actuation of wheels, but rather on the nonholonomic constraints imposed on the motion of the system by the rolling without slipping of the idler wheels of each module on the supporting plane. This results in a snake-like motion of the VGT assembly, which is not too far, at least in principle, from certain modes of actual snake locomotion (Hirose [1993]). Both the shape and the configuration of the VGT assembly can be described by elements of the Special Euclidean group  $SE(2)$ , the group of rigid motions on the plane. A system like the VGT assembly constitutes a kinematic chain evolving on this matrix Lie group, with the corresponding velocities given by elements of the Lie algebra of  $SE(2)$ . Of these velocities, the shape variations can be considered as the controls of the system and they are referred to as *shape controls*. The nonholonomic constraints allow us to express the global motion of the VGT assembly as a function of the shape and of the shape controls and to formulate motion control strategies under periodic shape controls.

This situation can be generalized to kinematic chains evolving on an arbitrary (matrix) Lie group  $G$  under a certain class of nonholonomic constraints. In particular, we are interested in groups with a real finite-dimensional non-abelian Lie algebra  $\mathcal{G}$  (of dimension  $n$ ) and  $(\ell - 1)$ -module kinematic chains evolving on them, where the constraints force the velocities of the system to lie in a subspace of  $\mathcal{G}$ , which is not a subalgebra of  $\mathcal{G}$  but which generates the whole algebra  $\mathcal{G}$  under Lie bracketing. We refer to systems of this type as  $G$ -snakes and observe that they possess an interesting geometric structure: When  $\ell = n$  and the codimension of the constraints is one, the configuration and shape spaces of the system specify a principal fiber bundle (Bleeker [1981];

Nomizu [1956]) and the nonholonomic constraints specify a (partial) connection on it, at least away from certain configurations which we call *nonholonomic singularities* (higher codimension cases will be treated elsewhere).

In section 2 of this paper, we consider an  $(\ell - 1)$ -module kinematic chain evolving on an  $n$ -dimensional Lie group. The Wei-Norman representation of  $G$  (Wei & Norman [1964]), which expresses each element of the group as a product of the one-parameter subgroups of  $G$ , and the notion of the adjoint action of  $G$  on  $\mathcal{G}$  allow us to express in a compact form how the motion of each module of the kinematic chain relates to that of the other modules and to the global motion of the system and how this latter becomes a function of just the shape and the shape controls because of the nonholonomic constraints. We show that the configuration and shape spaces of the  $G$ -snake specify a principal fiber bundle and that the nonholonomic constraints specify a connection on it.

In section 3 we focus on 2-module  $G$ -snakes ( $\ell = 3$ ) evolving on 3-dimensional Lie groups ( $n = 3$ ). In particular, we examine, apart from  $SE(2)$ , the Heisenberg group  $H(3)$ , the Special Orthogonal group  $SO(3)$  and the Special Linear group  $SL(2)$ . We derive the corresponding Wei-Norman representation, the system kinematics, the connection and specify the nonholonomic singularities in each case.

In section 4 we discuss possible further extensions of this work. For reasons having to do with ease of exposition, we limit ourselves to matrix Lie groups in this paper. Extensions to arbitrary Lie groups are easy.

## 2 Nonholonomic Kinematic Chains on Lie Groups

In section 2.1 we discuss the Wei-Norman representation of curves in  $n$ -dimensional Lie groups and in section 2.2 we derive the kinematics of the  $(\ell - 1)$ -module kinematic chain. In section 2.3 we examine the geometric structure of the chain kinematics when nonholonomic constraints are present, using the theory of connections on principal fiber bundles.

### 2.1 The Wei-Norman Representation of Lie Groups

Consider a left-invariant dynamical system on a matrix Lie group  $G$  with  $n$ -dimensional Lie algebra  $\mathcal{G}$ . For  $g \in G$ , the *left translation* by  $g$  is defined as the map  $L_g : G \rightarrow G : h \mapsto gh$ , for  $h \in G$ . If  $e$  is the identity of  $G$ , then  $T_e L_g$  is the tangent of the map  $L_g$  at  $e$ . Consider a curve  $g(\cdot) \subset G$ . Then, there exists a curve  $\xi(\cdot) \in \mathcal{G}$  such that:

$$\dot{g} = T_e L_g \cdot \xi = g \cdot \xi . \quad (1)$$

Let  $\{\mathcal{A}_i, i = 1, \dots, n\}$  be a basis of  $\mathcal{G}$  and let  $[\cdot, \cdot]$  be the usual Lie bracket on  $\mathcal{G}$  defined by:  $[\mathcal{A}_i, \mathcal{A}_j] = \mathcal{A}_i \mathcal{A}_j - \mathcal{A}_j \mathcal{A}_i$ . Then, there exist constants  $\Gamma_{i,j}^k$ , called *structure constants*, such that:

$$[\mathcal{A}_i, \mathcal{A}_j] = \sum_{k=1}^n \Gamma_{i,j}^k \mathcal{A}_k, \quad i, j = 1, \dots, n. \quad (2)$$

Let  $\mathcal{G}^*$  be the dual space of  $\mathcal{G}$ , i.e. the space of linear functions from  $\mathcal{G}$  to  $\mathbb{R}$ . Let  $\{\mathcal{A}_i^b, i = 1, \dots, n\}$  be the basis of  $\mathcal{G}^*$  such that

$$\mathcal{A}_i^b(\mathcal{A}_j) = \delta_i^j, \quad i, j = 1, \dots, n, \quad (3)$$

where  $\delta_i^j$  is the Kronecker symbol. Then the curve  $\xi(\cdot) \subset \mathcal{G}$  can be represented as:

$$\xi = \sum_{i=1}^n \xi_i \mathcal{A}_i = \sum_{i=1}^n \mathcal{A}_i^b(\xi) \mathcal{A}_i. \quad (4)$$

**Proposition 2.1.1** (Wei & Norman [1964])

Let  $g(0) = e$ , the identity of  $G$  and let  $g(t)$  be the solution of (1). Then, locally around  $t = 0$ ,  $g$  is of the form:

$$g(t) = e^{\gamma_1(t) \mathcal{A}_1} e^{\gamma_2(t) \mathcal{A}_2} \dots e^{\gamma_n(t) \mathcal{A}_n}, \quad (5)$$

where the coefficients  $\gamma_i$  are determined by differentiating (5) and using (1). Then:

$$\begin{pmatrix} \dot{\gamma}_1 \\ \vdots \\ \dot{\gamma}_n \end{pmatrix} = M(\gamma_1, \dots, \gamma_n) \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}. \quad (6)$$

The matrix  $M$  is analytic in  $\gamma$  and depends only on the Lie algebra  $\mathcal{G}$  and its structure constants in the given basis. If  $\mathcal{G}$  is solvable, then there exists a basis of  $\mathcal{G}$  and an ordering of this basis, for which (5) is global. Then the  $\gamma_i$ 's can be found by quadratures. ■

For  $g \in G$ ,  $\xi \in \mathcal{G}$ , define the *adjoint action* of  $G$  on  $\mathcal{G}$  denoted  $Ad_g : \mathcal{G} \rightarrow \mathcal{G}$  by:

$$Ad_g \xi \stackrel{\text{def}}{=} g \xi g^{-1}. \quad (7)$$

From (4) we have:

$$Ad_g \xi = \sum_{i=1}^3 \xi_i Ad_g \mathcal{A}_i = \sum_{i=1}^3 \mathcal{A}_i^b(\xi) Ad_g \mathcal{A}_i. \quad (8)$$

From (5) we have:

$$Ad_g \mathcal{A}_i = g \mathcal{A}_i g^{-1} = e^{\gamma_1 \mathcal{A}_1} \dots e^{\gamma_n \mathcal{A}_n} \mathcal{A}_i e^{-\gamma_n \mathcal{A}_n} \dots e^{-\gamma_1 \mathcal{A}_1} . \quad (9)$$

Equation (9) can be made more explicit by the Baker–Campbell–Hausdorff formula (Wei & Norman [1964]), which for  $x, y \in \mathcal{G}$  states that:

$$e^x y e^{-x} = y + [x, y] + \frac{1}{2!} [x, [x, y]] + \dots . \quad (10)$$

## 2.2 The $(\ell - 1)$ -module Kinematic Chain

We consider a dynamical system that evolves on the Cartesian product  $Q = \underbrace{G \times \dots \times G}_{\ell \text{ times}}$ .

Its trajectory is a curve  $g(\cdot) = (g_1(\cdot), \dots, g_\ell(\cdot)) \subset Q$ . On each copy of  $G$ , the system traces a curve  $g_i(\cdot) \subset G$ , such that

$$\dot{g}_i = T_e L_{g_i} \cdot \xi_i = g_i \cdot \xi_i , \quad i = 1, \dots, \ell, \quad (11)$$

where  $\xi_i(\cdot) \in \mathcal{G}$ ,  $i = 1, \dots, \ell$ . We think of the  $g_i$ 's as the *nodes* of a kinematic chain.

Let the instantaneous *shape* of the kinematic chain be given by the  $(\ell - 1)$ -tuple  $(g_{1,2}, g_{2,3}, \dots, g_{\ell-1,\ell}) \in \underbrace{G \times \dots \times G}_{(\ell-1) \text{ times}}$ , where

$$g_{i,i+1} = g_i^{-1} g_{i+1} , \quad i = 1, \dots, \ell - 1 . \quad (12)$$

A pair of adjacent nodes of the chain constitutes a *module*. The  $g_{i,i+1}$ 's can be regarded as the shapes of the modules of an  $(\ell - 1)$ -module kinematic chain. We consider the corresponding curves  $\xi_{i,i+1} \subset \mathcal{G}$ , specified by (1) as the controls of the system (11). We refer to them as the *shape controls*:

$$\dot{g}_{i,i+1} = T_e L_{g_{i,i+1}} \cdot \xi_{i,i+1} = g_{i,i+1} \cdot \xi_{i,i+1} , \quad i = 1, \dots, \ell - 1 . \quad (13)$$

For future reference, define also:

$$g_{i,j} = g_i^{-1} g_j = g_{i,i+1} \dots g_{j-1,j} , \quad i \leq j \quad (14)$$

and

$$\dot{g}_{i,j} = T_e L_{g_{i,j}} \cdot \xi_{i,j} = g_{i,j} \cdot \xi_{i,j} , \quad i \leq j . \quad (15)$$



We can think of the  $\xi_i$ 's as characterizing the global motion of the  $G$ -snake system with respect to some global coordinate system, while the  $\xi_{i,j}$ 's capture the relative motion (or shape variation) of nodes  $i$  and  $j$ .

From (11), (12) and (13) we get:

$$\xi_i = \xi_{i-1,i} + Ad_{(g_{i-1}^{-1}g_i)^{-1}}\xi_{i-1}, \quad i = 2, \dots, \ell. \quad (16)$$

Applying (16) iteratively we can express any  $\xi_i$  as a function of  $\xi_1$  and of the shape controls  $\xi_{1,2}, \dots, \xi_{i-1,i}$  as follows:

$$\xi_i = \xi_{i-1,i} + Ad_{(g_{i-1}^{-1}g_i)^{-1}}\xi_{i-2,i-1} + \dots + Ad_{(g_2^{-1}g_i)^{-1}}\xi_{1,2} + Ad_{(g_1^{-1}g_i)^{-1}}\xi_1. \quad (17)$$

Using (4) we get:

$$\begin{aligned} \xi_i = & \sum_{j=1}^n \mathcal{A}_j^b(\xi_{i-1,i})\mathcal{A}_j + \sum_{j=1}^n \mathcal{A}_j^b(\xi_{i-2,i-1})Ad_{(g_{i-1}^{-1}g_i)^{-1}}\mathcal{A}_j \\ & + \dots + \sum_{j=1}^n \mathcal{A}_j^b(\xi_{1,2})Ad_{(g_2^{-1}g_i)^{-1}}\mathcal{A}_j + \sum_{j=1}^n \mathcal{A}_j^b(\xi_1)Ad_{(g_1^{-1}g_i)^{-1}}\mathcal{A}_j. \end{aligned} \quad (18)$$

### 2.3 Nonholonomic Constraints and Connections on Principal Fiber Bundles

In this section we consider nonholonomic constraints acting on the  $G$ -snake and we show that they specify a connection on the principal fiber bundle associated to our problem.

**Codimension 1 Constraint Hypothesis:** Assume that the evolution of system (11) on each copy of  $G$  is constrained to lie on an  $(n-1)$ -dimensional subspace  $h$  of the Lie algebra  $\mathcal{G}$ , where  $h$  is not a subalgebra of  $\mathcal{G}$ , i.e.

$$\xi_i \in h, \quad i = 1, \dots, \ell. \quad (19)$$

Then, for some  $\mathcal{A}_\kappa^b \in \mathcal{G}^*$  (not necessarily an element of the basis  $\{\mathcal{A}_i^b, i = 1, \dots, n\}$ ) we have:

$$h = \text{Ker}(\mathcal{A}_\kappa^b). \quad (20)$$

The constraints (19) can be expressed as:

$$\mathcal{A}_\kappa^b(\xi_i) = 0, \quad i = 1, \dots, \ell. \quad (21)$$

The constraints (21) are linear in the components of  $\xi_1$  and those of the shape controls  $\xi_{i,j}$ . This can be made explicit by defining the composite *velocity vector* of the kinematic chain:

$$\begin{aligned}\Xi &\stackrel{\text{def}}{=} (\xi_1^1 \dots \xi_n^1 \xi_1^{1,2} \dots \xi_n^{\ell-1,\ell})^\top \\ &= (\mathcal{A}_1^b(\xi_1) \dots \mathcal{A}_n^b(\xi_1) \mathcal{A}_1^b(\xi_{1,2}) \dots \mathcal{A}_n^b(\xi_{\ell-1,\ell}))^\top.\end{aligned}$$

**Proposition 2.3.1**

The  $\ell$  nonholonomic constraints (21) can be written in matrix form as:

$$A(g_{1,2}, \dots, g_{\ell-1,\ell}) \Xi = 0, \quad (22)$$

where  $A$  is a function of only the shape of the system and is a block lower triangular  $\ell \times n\ell$  matrix of maximal rank of the form

$$A = \begin{pmatrix} *_{1,1} & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ *_{1,2} & *_{2,2} & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \ddots & 0 & \dots & 0 & 0 \\ *_{1,i} & *_{2,i} & \dots & *_{i-1,i} & *_{i,i} & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \dots & \ddots & 0 \\ *_{1,\ell} & *_{2,\ell} & \dots & *_{i-1,\ell} & *_{i,\ell} & \dots & *_{\ell-1,\ell} & *_{\ell,\ell} \end{pmatrix}, \quad (23)$$

with the  $1 \times n$  block  $*_{p,q}$ , defined for  $p \leq q$  as:

$$*_{p,q} = \left( \mathcal{A}_\kappa^b(Ad_{(g_p^{-1}g_q)^{-1}}\mathcal{A}_1) \dots \mathcal{A}_\kappa^b(Ad_{(g_p^{-1}g_q)^{-1}}\mathcal{A}_n) \right).$$

**Proof**

From (18) and (21) :

$$\begin{aligned}\mathcal{A}_\kappa^b(\xi_1) &= \sum_{j=1}^n \mathcal{A}_j^b(\xi_1) \mathcal{A}_\kappa^b(\mathcal{A}_j) = 0, \\ \mathcal{A}_\kappa^b(\xi_i) &= \sum_{j=1}^n \mathcal{A}_j^b(\xi_{i-1,i}) \mathcal{A}_\kappa^b(\mathcal{A}_j) + \sum_{j=1}^n \mathcal{A}_j^b(\xi_{i-2,i-1}) \mathcal{A}_\kappa^b(Ad_{(g_{i-1}^{-1}g_i)^{-1}}\mathcal{A}_j) \\ &\quad + \dots + \sum_{j=1}^n \mathcal{A}_j^b(\xi_{1,2}) \mathcal{A}_\kappa^b(Ad_{(g_2^{-1}g_1)^{-1}}\mathcal{A}_j) + \sum_{j=1}^n \mathcal{A}_j^b(\xi_1) \mathcal{A}_\kappa^b(Ad_{(g_1^{-1}g_i)^{-1}}\mathcal{A}_j), \\ &\quad i = 2, \dots, \ell.\end{aligned} \quad (24)$$

The diagonal blocks  $*_{p,p}$  of  $A$  have the form  $\begin{pmatrix} \mathcal{A}_\kappa^b(\mathcal{A}_1) & \cdots & \mathcal{A}_\kappa^b(\mathcal{A}_n) \end{pmatrix}$ , therefore they contain at least one non-zero constant term. Thus  $A$  has always maximal rank. ■

### Proposition 2.3.2

Assume  $\ell \geq n$ . Partition  $\Xi$  as  $(\Xi_1 \ \Xi_2)$ , with  $\Xi_2$  an  $\ell$ -dimensional vector containing the components of  $\xi_1$  (and possibly some components of shape controls), while  $\Xi_1$  is an  $(n-1)\ell$ -dimensional vector containing *only* components of shape controls. Let the corresponding partition of  $A$  be  $(A_1 \ A_2)$ , with  $A_1$  a  $(n-1)\ell \times \ell$  matrix and  $A_2$  a locally invertible  $\ell \times \ell$  matrix. Then from (22) :

$$\Xi_2 = -A_2^{-1}(g_{1,2}, \dots, g_{\ell-1,\ell})A_1(g_{1,2}, \dots, g_{\ell-1,\ell})\Xi_1. \quad (25)$$

### Proof

Follows from the smooth dependence of  $A$  on the shape variables and the maximal rank property of Proposition 2.3.1. ■

The physical significance of this result is that, if the global motion of the  $(\ell-1)$ -module system is characterized by the global motion of its first module (i.e. by  $\xi_1$ ), then variations of the shape controls (at least those which are elements of  $\Xi_1$ ) induce a global motion of the system.

Configurations where  $A_2$  is singular will be called *nonholonomic singularities*.

### Definition 2.3.3 (Principal Fiber Bundle (Nomizu [1956]) )

Let  $S$  be a differentiable manifold and  $G$  a Lie group. A differentiable manifold  $Q$  is called a (differentiable) *principal fiber bundle* if the following conditions are satisfied:

1)  $G$  acts on  $Q$  to the left, freely and differentiably:

$$\Phi : G \times Q \rightarrow Q : (g, q) \mapsto g \cdot q \stackrel{\text{def}}{=} \Phi_g \cdot q.$$

2)  $S$  is the quotient space of  $Q$  by the equivalence relation induced by  $G$ , i.e.  $S = Q/G$  and the canonical projection  $\pi : Q \rightarrow S$  is differentiable.

3)  $Q$  is locally trivial, i.e. every point  $s \in S$  has a neighborhood  $U$  such that  $\pi^{-1}(U) \subset Q$  is isomorphic with  $U \times G$ , in the sense that  $q \in \pi^{-1}(U) \mapsto (\pi(q), \phi(q)) \in U \times G$  is a diffeomorphism such that  $\phi : \pi^{-1}(U) \rightarrow G$  satisfies  $\phi(g \cdot q) = g\phi(q), \forall g \in G$ .

For  $s \in S$ , the *fiber over  $s$*  is a closed submanifold of  $Q$  which is differentiably isomorphic with  $G$ . For any  $q \in Q$ , the *fiber through  $q$*  is the fiber over  $s = \pi(q)$ . When  $Q = S \times G$ , then  $Q$  is said to be a *trivial* principal fiber bundle.

■

Consider now the manifolds  $Q$  and  $S$  defined in section 2.2 and the canonical projection  $\pi : Q \rightarrow S$  defined by equation (12), i.e.

$$\pi(g_1, \dots, g_\ell) \stackrel{\text{def}}{=} (g_1^{-1}g_2, \dots, g_{\ell-1}^{-1}g_\ell) = (g_{1,2}, \dots, g_{\ell-1,\ell}). \quad (26)$$

We can show that the quadruple  $(Q, S, \pi, G)$ , together with the action  $\Phi$  of  $G$  on  $Q$  defined by

$$\Phi : G \times Q \rightarrow Q : (g, q) = (g, (g_1, \dots, g_\ell)) \mapsto g \cdot q = (gg_1, \dots, gg_\ell), \quad (27)$$

meets the requirements of Definition 2.3.3, and therefore  $(Q, S, \pi, G)$  is a trivial principal fiber bundle. Indeed,  $\pi$  is differentiable and its differential is

$$\pi_{*q} : T_q Q \rightarrow T_{\pi(q)} S : (g_1 \xi_1, \dots, g_\ell \xi_\ell) \mapsto (g_{1,2} \xi_{1,2}, \dots, g_{\ell-1,\ell} \xi_{\ell-1,\ell}), \quad (28)$$

where the  $\xi_{i-1,i}$  are given by (16) :

$$\xi_{i-1,i} = \xi_i - \text{Ad}_{(g_{i-1}^{-1}g_i)^{-1}} \xi_{i-1}, \quad i = 2, \dots, \ell. \quad (29)$$

**Definition 2.3.4** (Connection on a Principal Fiber Bundle (Nomizu [1956]))

Let  $(Q, S, \pi, G)$  be a principal fiber bundle. A *connection* on the principal fiber bundle is a choice of a tangent subspace  $H_q \subset T_q Q$  at each point  $q \in Q$  (horizontal subspace) such that, if  $V_q \stackrel{\text{def}}{=} \{v \in T_q Q \mid \pi_{*q}(v) = 0\}$  is the subspace of  $T_q Q$  tangent to the fiber through  $q$  (vertical subspace), we have:

- 1)  $T_q Q = H_q \oplus V_q$ .
- 2) For every  $g \in G$  and  $q \in Q$ ,  $T_q \Phi_g \cdot H_q = H_{g \cdot q}$ .
- 3)  $H_q$  depends differentiably on  $q$ .

■

### Proposition 2.3.5

Away from nonholonomic singularities and when  $\ell = n$ , the nonholonomic constraints (21) specify a connection on the principal fiber bundle  $(Q, S, \pi, G)$ , with the horizontal subspace defined as follows:

$$\begin{aligned} H_q &= \{v \in T_q Q \mid v = (g_1 \xi_1, \dots, g_\ell \xi_\ell) \text{ and } \xi_i \in h\} \\ &= \{v \in T_q Q \mid v = (g_1 \xi_1, \dots, g_\ell \xi_\ell) \text{ and } \Xi_2 = -A_2^{-1}(\pi(q))A_1(\pi(q))\Xi_1\} , \end{aligned} \quad (30)$$

where  $\Xi_1 = (\xi_1^{1,2} \dots \xi_n^{1,2} \xi_1^{2,3} \dots \xi_n^{\ell-1,\ell})^\top$  and  $\Xi_2 = (\xi_1^1 \dots \xi_n^1)^\top$ .

### Proof

Due to the left-invariance of our system,  $T_q Q = \{(g_1 \xi_1, \dots, g_\ell \xi_\ell) \mid \xi_i \in \mathcal{G}\}$ . The vertical subspace is (from (27) – (29))

$$\begin{aligned} V_q &= \{v \in T_q Q \mid \pi_{*q}(\xi_j) = 0\} \\ &= \{(g_1 \xi_1, \dots, g_\ell \xi_\ell) \mid (g_{1,2} \xi_{1,2}, \dots, g_{\ell-1,\ell} \xi_{\ell-1,\ell}) = 0\} \\ &= \{(g_1 \xi_1, \dots, g_\ell \xi_\ell) \mid \xi_{1,2} = \dots = \xi_{\ell-1,\ell} = 0\} \\ &= \{(g_1 \xi_1, \dots, g_\ell \xi_\ell) \mid \xi_i = Ad_{(g_1^{-1} g_i)^{-1}} \xi_1, i = 2, \dots, \ell\} . \end{aligned} \quad (31)$$

Physically, the vertical subspace contains all infinitesimal motions of the kinematic chain that do not alter its shape.

To show property (1) of Definition 2.3.4, we first prove that  $H_q \cap V_q = \{0\}$  and then that  $\dim(T_q Q) = \dim(H_q) + \dim(V_q)$ .

To show  $H_q \cap V_q = \{0\}$ , assume that there exists a non-trivial  $v = q \cdot \xi \in H_q \cap V_q$ . By the definition of  $V_q$ , the corresponding shape controls are zero. Thus  $\Xi_1 = 0$  and, by the definition of  $H_q$ , also  $\Xi_2 = 0$ . But then  $\xi_1 = 0$  and from (31) also  $\xi_i = 0$ ,  $i = 2, \dots, \ell$ . Thus  $\xi = 0$ . Thus  $H_q \cap V_q = \{0\}$ . Now observe that, away from the nonholonomic singularities  $\dim(H_q) = n\ell - \ell$ . Further,  $\dim(V_q) = n$ . So, when  $\ell = n$ ,  $\dim(H_q \oplus V_q) = (n\ell - \ell) + n = (n^2 - n) + n = n^2 = \dim(T_q Q)$ . It follows that  $H_q \oplus V_q = T_q Q$ .

To show property (2) of Definition 2.3.4, consider  $T_q \Phi_g \cdot H_q = g \cdot H_q = g \cdot \{(g_1 \xi_1, \dots, g_\ell \xi_\ell) \mid \xi_i \in h\} \stackrel{\text{def}}{=} \{(gg_1 \xi_1, \dots, gg_\ell \xi_\ell) \mid \xi_i \in h\}$  and  $H_{g \cdot q} = \{v \in T_{g \cdot q} Q \mid v = (g \cdot q) \cdot (\xi_1, \dots, \xi_\ell) \text{ and } \xi_i \in h\} = \{(gg_1 \xi_1, \dots, gg_\ell \xi_\ell) \mid \xi_i \in h\}$ . Then, obviously,  $T_q \Phi_g \cdot H_q = H_{g \cdot q}$ .

Property (3) of Definition 2.3.4 is immediate from the smooth dependence of  $A$  on the shape and from the left-invariance of our system. ■

### 3 Three-dimensional Nonholonomic Lie Groups

Here we specialize the results of the previous section to kinematic chains on Lie Groups with 3-dimensional real non-abelian Lie algebras ( $n = 3$ ). In section 3.1 we consider the Special Euclidean group  $SE(2)$ , in section 3.2 the Heisenberg group  $H(3)$ , in section 3.3 the Special Orthogonal group  $SO(3)$  and in section 3.4 the Special Linear group  $SL(2)$ .

We study 2-module ( $\ell = n = 3$ ) kinematic chains on each of these groups by deriving their Wei–Norman representation and by defining the partial connection on the corresponding principal fiber bundle.

Let  $G$  be one of the above four matrix Lie groups and  $\mathcal{G}$  be the corresponding Lie algebra. Consider the system (2.1) on  $G$ :

$$\dot{g} = T_e L_g \cdot \xi = g \cdot \xi , \quad (1)$$

with  $g \in G$  and  $\xi \in \mathcal{G}$ . From Proposition 2.1.1, any  $g \in G$  has a *local* Wei–Norman representation of the form

$$g(t) = e^{\gamma_1(t)A_1} e^{\gamma_2(t)A_2} e^{\gamma_3(t)A_3} . \quad (2)$$

Consider now the 2-module kinematic chain on  $G$ . From the system kinematics (equations (2.11) – (2.18)) we have:

$$\begin{aligned} g_2 &= g_1 g_{1,2} , \\ g_3 &= g_2 g_{2,3} = g_1 g_{1,2} g_{2,3} , \\ g_{1,3} &= g_{1,2} g_{2,3} . \end{aligned} \quad (3)$$

From (2.16) we get for the corresponding velocities:

$$\begin{aligned} \xi_2 &= \xi_{1,2} + Ad_{(g_1^{-1}g_2)^{-1}} \xi_1 , \\ \xi_3 &= \xi_{2,3} + Ad_{(g_2^{-1}g_3)^{-1}} \xi_2 = \xi_{2,3} + Ad_{(g_2^{-1}g_3)^{-1}} \xi_{1,2} + Ad_{(g_1^{-1}g_3)^{-1}} \xi_1 . \end{aligned} \quad (4)$$

Assume that the evolution of system (1) on each copy of  $G$  is constrained to lie on a 2-dimensional subspace  $h$  of the Lie algebra  $\mathcal{G}$ , where  $h$  is not a subalgebra of  $\mathcal{G}$ . Define

$$\Xi_1 = (\xi_1^{1,2} \ \xi_2^{1,2} \ \xi_3^{1,2} \ \xi_1^{2,3} \ \xi_2^{2,3} \ \xi_3^{2,3})^\top \quad \text{and} \quad \Xi_2 = (\xi_1^1 \ \xi_2^1 \ \xi_3^1)^\top .$$

Proposition 2.3.1 holds with  $\Xi = (\Xi_1 \ \Xi_2)$ . From Proposition 2.3.2 we conclude that the global velocity of the 2-module kinematic chain, as it is characterized by  $\xi_1$ , can be expressed as a function of only the shape variables  $g_{1,2}, g_{2,3}$  and shape controls  $\xi_{1,2}, \xi_{2,3}$  of the assembly:

$$\Xi_2 = -A_2^{-1}(g_{1,2}, g_{2,3}) A_1(g_{1,2}, g_{2,3}) \Xi_1 . \quad (5)$$

From Proposition 2.3.5, equation (5) defines (away from the singularities of  $A_2$ ) a connection on the trivial principal bundle  $(S \times G, S, \pi, G)$  with  $S = G \times G$  and with horizontal subspace:

$$\begin{aligned} H_q &= \{v \in T_q Q \mid v = (g_1 \xi_1, g_2 \xi_2, g_3 \xi_3) \text{ and } \xi_i \in h\} \\ &= \{v \in T_q Q \mid v = (g_1 \xi_1, g_2 \xi_2, g_3 \xi_3) \text{ and } \Xi_2 = -A_2^{-1}(g_{1,2}, g_{2,3}) A_1(g_{1,2}, g_{2,3}) \Xi_1 \} . \end{aligned} \quad (6)$$

Subsequently, we will derive explicitly the Wei–Norman representation for each of the Lie groups mentioned earlier and we will define the connection (6) for specific 2-dimensional subspaces  $h$  of  $\mathcal{G}$ . In (Vershik & Gershkovich [1994]) the authors present a result showing that, for each of  $H(3)$ ,  $SO(3)$  and  $SE(2)$ , all 2-dimensional subspaces  $h$  of  $\mathcal{G}$ , which are not subalgebras, are isomorphic and can be represented by  $h = \text{sp}\{\mathcal{A}_1, \mathcal{A}_2\}$  in the basis of  $\mathcal{G}$  specified in the following sections. For  $SL(2)$ , there are 2 classes of such isomorphic subspaces that can be represented, respectively, by  $h = \text{sp}\{\mathcal{A}_1, \mathcal{A}_2\}$  and by  $h = \text{sp}\{\mathcal{A}_3, \mathcal{A}_1 + \mathcal{A}_2\}$ . Therefore, only nonholonomic constraints corresponding to these subspaces of  $\mathcal{G}$  will be considered here.

Our main purpose in this section is to set the stage for a deeper understanding of this novel class of kinematic chains, by cataloguing the low-dimensional possibilities. One case, corresponding to  $SE(2)$  has already found a concrete mechanical realization (Krishnaprasad & Tsakiris [1994]). Others might follow, for instance, there are possible connections between  $SO(3)$ -snakes and the kinematics of long chain molecules (Karplus & McCammon [1986]).

### 3.1 Nonholonomic Kinematic Chains on the Special Euclidean Group $SE(2)$

Let  $G = SE(2)$  be the Special Euclidean group of rigid motions on the plane and  $\mathcal{G} = \mathfrak{se}(2)$  be the corresponding algebra with the following basis:

$$\mathcal{A}_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathcal{A}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathcal{A}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} . \quad (7)$$

Then:

$$[\mathcal{A}_1, \mathcal{A}_2] = \mathcal{A}_3, [\mathcal{A}_1, \mathcal{A}_3] = -\mathcal{A}_2, [\mathcal{A}_2, \mathcal{A}_3] = 0 . \quad (8)$$

The algebra  $\mathcal{G}$  is solvable and, from Proposition 2.1.1, any  $g \in G$  has a *global* Wei–Norman representation of the form (2), with

$$\begin{pmatrix} \dot{\gamma}_1 \\ \dot{\gamma}_2 \\ \dot{\gamma}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \gamma_3 & 1 & 0 \\ -\gamma_2 & 0 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}. \quad (9)$$

Equation (9) is solvable by quadratures:

$$\begin{aligned} \gamma_1(t) &= \gamma_1(0) + \int_0^t \xi_1(\tau) d\tau, \\ \gamma_2(t) &= \gamma_2(0) \cos \left( \int_0^t \xi_1(\sigma) d\sigma \right) + \gamma_3(0) \sin \left( \int_0^t \xi_1(\sigma) d\sigma \right) \\ &\quad + \int_0^t \xi_2(\tau) \cos \left( \int_\tau^t \xi_1(\sigma) d\sigma \right) d\tau + \int_0^t \xi_3(\tau) \sin \left( \int_\tau^t \xi_1(\sigma) d\sigma \right) d\tau, \\ \gamma_3(t) &= -\gamma_2(0) \sin \left( \int_0^t \xi_1(\sigma) d\sigma \right) + \gamma_3(0) \cos \left( \int_0^t \xi_1(\sigma) d\sigma \right) \\ &\quad - \int_0^t \xi_2(\tau) \sin \left( \int_\tau^t \xi_1(\sigma) d\sigma \right) d\tau + \int_0^t \xi_3(\tau) \cos \left( \int_\tau^t \xi_1(\sigma) d\sigma \right) d\tau. \end{aligned} \quad (10)$$

From (2.9), (9) and (10) we can compute  $Ad_{g^{-1}}\mathcal{A}_i$ :

$$\begin{aligned} Ad_{g^{-1}}\mathcal{A}_1 &= \mathcal{A}_1 - \gamma_3\mathcal{A}_2 + \gamma_2\mathcal{A}_3, \\ Ad_{g^{-1}}\mathcal{A}_2 &= \cos \gamma_1\mathcal{A}_2 - \sin \gamma_1\mathcal{A}_3, \\ Ad_{g^{-1}}\mathcal{A}_3 &= \sin \gamma_1\mathcal{A}_2 + \cos \gamma_1\mathcal{A}_3. \end{aligned} \quad (11)$$

Consider now the 2-module kinematic chain on  $SE(2)$ , ( $\ell = 3$ ). A concrete mechanical realization of such a system is the Variable Geometry Truss assembly mentioned in section 1, which appears in fig. 3.1. The system kinematics of equations (3), (4) apply.

From (8) we can see that there are two possible 2-dimensional subspaces  $h$  of  $\mathcal{G}$  that can generate the whole algebra under Lie bracketing:

$$h_3 = \text{sp}\{\mathcal{A}_1, \mathcal{A}_2\} = \text{Ker}(\mathcal{A}_3^b) \text{ and } h_2 = \text{sp}\{\mathcal{A}_1, \mathcal{A}_3\} = \text{Ker}(\mathcal{A}_2^b). \quad (12)$$



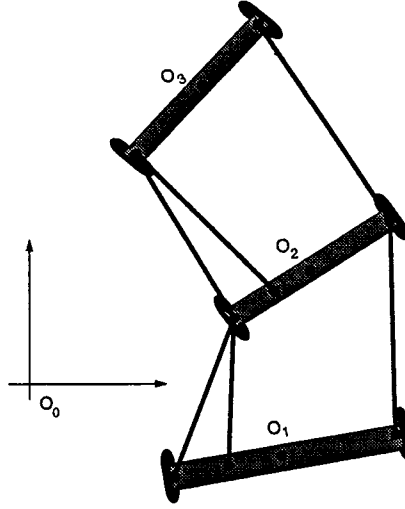


Fig. 3.1

Subsequently we will consider only  $h_2$  (which is exactly the case of the system in fig. 3.1). The nonholonomic constraints  $\xi_i \in h_2$  can, then, be expressed as:

$$\mathcal{A}_2^b(\xi_i) = 0, \quad i = 1, 2, 3. \quad (13)$$

Equation (5) holds with  $\Xi_1$  and  $\Xi_2$  defined as above and with:

$$\begin{aligned} A_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \mathcal{A}_2^b(Ad_{(g_2^{-1}g_3)}^{-1}\mathcal{A}_1) & \mathcal{A}_2^b(Ad_{(g_2^{-1}g_3)}^{-1}\mathcal{A}_2) & \mathcal{A}_2^b(Ad_{(g_2^{-1}g_3)}^{-1}\mathcal{A}_3) & 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\gamma_3^{2,3} & \cos \gamma_1^{2,3} & \sin \gamma_1^{2,3} & 0 & 1 & 0 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} A_2 &= \begin{pmatrix} 0 & 1 & 0 \\ \mathcal{A}_2^b(Ad_{(g_1^{-1}g_2)}^{-1}\mathcal{A}_1) & \mathcal{A}_2^b(Ad_{(g_1^{-1}g_2)}^{-1}\mathcal{A}_2) & \mathcal{A}_2^b(Ad_{(g_1^{-1}g_2)}^{-1}\mathcal{A}_3) \\ \mathcal{A}_2^b(Ad_{(g_1^{-1}g_3)}^{-1}\mathcal{A}_1) & \mathcal{A}_2^b(Ad_{(g_1^{-1}g_3)}^{-1}\mathcal{A}_2) & \mathcal{A}_2^b(Ad_{(g_1^{-1}g_3)}^{-1}\mathcal{A}_3) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 0 \\ -\gamma_3^{1,2} & \cos \gamma_1^{1,2} & \sin \gamma_1^{1,2} \\ -\gamma_3^{1,3} & \cos \gamma_1^{1,3} & \sin \gamma_1^{1,3} \end{pmatrix}. \end{aligned}$$

The nonholonomic singularities of the system are the configurations where:

$$\begin{aligned} \det(A_2) &= \mathcal{A}_2^b(Ad_{(g_1^{-1}g_3)}^{-1}\mathcal{A}_1)\mathcal{A}_2^b(Ad_{(g_1^{-1}g_2)}^{-1}\mathcal{A}_3) - \mathcal{A}_2^b(Ad_{(g_1^{-1}g_2)}^{-1}\mathcal{A}_1)\mathcal{A}_2^b(Ad_{(g_1^{-1}g_3)}^{-1}\mathcal{A}_3) \\ &= -\gamma_3^{1,3} \sin \gamma_1^{1,2} + \gamma_3^{1,2} \sin \gamma_1^{1,3} = 0. \end{aligned}$$

See (Krishnaprasad & Tsakiris [1994]) for the mechanical interpretation of the nonholonomic singularities. Away from those, equation (6) specifies the connection corresponding to this system. Under periodic shape controls  $\xi_{1,2}, \xi_{2,3}$ , the system translates when  $\gamma_1^{1,2} = 0$  and rotates otherwise. This in turn is associated with the concept of geometric phase of the system, which is discussed in greater detail in (Krishnaprasad & Tsakiris [1994]). In the case of the translating system, the corresponding geometric phase is plotted in fig. 3.2 for one period of the shape controls and for varying amplitudes.

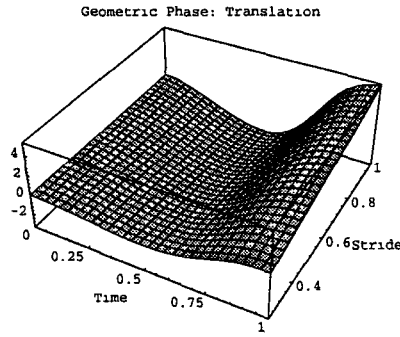


Fig. 3.2

### 3.2 Nonholonomic Kinematic Chains on the Heisenberg Group $H(3)$

Let  $G = H(3)$  be the Heisenberg group of real  $3 \times 3$  upper triangular matrices of the form  $\begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix}$  and let  $\mathcal{G} = h(3)$  be its algebra with the following basis:

$$\mathcal{A}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathcal{A}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \mathcal{A}_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (14)$$

Then:

$$[\mathcal{A}_1, \mathcal{A}_2] = \mathcal{A}_3, [\mathcal{A}_1, \mathcal{A}_3] = 0, [\mathcal{A}_2, \mathcal{A}_3] = 0. \quad (15)$$

The algebra  $\mathcal{G}$  is nilpotent (thus solvable) and, from Proposition 2.1.1, any  $g \in G$  has a *global* Wei–Norman representation of the form (2) with

$$\begin{pmatrix} \dot{\gamma}_1 \\ \dot{\gamma}_2 \\ \dot{\gamma}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\gamma_2 & 0 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}. \quad (16)$$

Equation (16) is solvable by quadratures:

$$\begin{aligned}
\gamma_1(t) &= \gamma_1(0) + \int_0^t \xi_1(\tau) d\tau , \\
\gamma_2(t) &= \gamma_2(0) + \int_0^t \xi_2(\tau) d\tau , \\
\gamma_3(t) &= \gamma_3(0) - \int_0^t \gamma_2(\tau) \xi_1(\tau) d\tau + \int_0^t \xi_3(\tau) d\tau \\
&= \gamma_3(0) - \gamma_2(0) \int_0^t \xi_1(\sigma) d\sigma - \int_0^t \xi_1(\tau) \left( \int_0^\tau \xi_2(\sigma) d\sigma \right) d\tau + \int_0^t \xi_3(\tau) d\tau .
\end{aligned} \tag{17}$$

From (2.9), (16) and (17) we can compute  $Ad_{g^{-1}}\mathcal{A}_i$  :

$$\begin{aligned}
Ad_{g^{-1}}\mathcal{A}_1 &= \mathcal{A}_1 + \gamma_2\mathcal{A}_3 , \\
Ad_{g^{-1}}\mathcal{A}_2 &= \mathcal{A}_2 - \gamma_1\mathcal{A}_3 , \\
Ad_{g^{-1}}\mathcal{A}_3 &= \mathcal{A}_3 .
\end{aligned} \tag{18}$$

Consider now the 2-module kinematic chain on  $G$ , ( $\ell = 3$ ). The system kinematics of equations (3), (4) apply.

From (15) we can see that there is only one possible 2-dimensional subspace  $h$  of  $\mathcal{G}$  that can generate the whole algebra under Lie bracketing:

$$h = \text{sp}\{\mathcal{A}_1, \mathcal{A}_2\} = \text{Ker}(\mathcal{A}_3^b) . \tag{19}$$

The nonholonomic constraints can, then, be expressed as:

$$\mathcal{A}_3^b(\xi_i) = 0, \quad i = 1, 2, 3 . \tag{20}$$

Equation (5) holds with  $\Xi_1$  and  $\Xi_2$  defined as above and with:

$$\begin{aligned}
A_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \mathcal{A}_3^b(Ad_{(g_2^{-1}g_3)^{-1}}\mathcal{A}_1) & \mathcal{A}_3^b(Ad_{(g_2^{-1}g_3)^{-1}}\mathcal{A}_2) & \mathcal{A}_3^b(Ad_{(g_2^{-1}g_3)^{-1}}\mathcal{A}_3) & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \gamma_2^{2,3} & -\gamma_1^{2,3} & 1 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
A_2 &= \begin{pmatrix} 0 & 0 & 1 \\ \mathcal{A}_3^b(Ad_{(g_1^{-1}g_2)^{-1}}\mathcal{A}_1) & \mathcal{A}_3^b(Ad_{(g_1^{-1}g_2)^{-1}}\mathcal{A}_2) & \mathcal{A}_3^b(Ad_{(g_1^{-1}g_2)^{-1}}\mathcal{A}_3) \\ \mathcal{A}_3^b(Ad_{(g_1^{-1}g_3)^{-1}}\mathcal{A}_1) & \mathcal{A}_3^b(Ad_{(g_1^{-1}g_3)^{-1}}\mathcal{A}_2) & \mathcal{A}_3^b(Ad_{(g_1^{-1}g_3)^{-1}}\mathcal{A}_3) \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 1 \\ \gamma_2^{1,2} & -\gamma_1^{1,2} & 1 \\ \gamma_2^{1,3} & -\gamma_1^{1,3} & 1 \end{pmatrix}.
\end{aligned}$$

The nonholonomic singularities of the system are the configurations where:

$$\begin{aligned}
\det(A_2) &= \mathcal{A}_3^b(Ad_{(g_1^{-1}g_2)^{-1}}\mathcal{A}_1)\mathcal{A}_3^b(Ad_{(g_1^{-1}g_3)^{-1}}\mathcal{A}_2) - \mathcal{A}_3^b(Ad_{(g_1^{-1}g_3)^{-1}}\mathcal{A}_1)\mathcal{A}_3^b(Ad_{(g_1^{-1}g_2)^{-1}}\mathcal{A}_2) \\
&= -\gamma_2^{1,2}\gamma_1^{1,3} + \gamma_1^{1,2}\gamma_2^{1,3} = -\gamma_2^{1,2}\gamma_1^{2,3} + \gamma_1^{1,2}\gamma_2^{2,3} = 0.
\end{aligned}$$

### 3.3 Nonholonomic Kinematic Chains on the Special Orthogonal Group SO(3)

Let  $G = SO(3)$  be the Special Orthogonal group of real orthogonal  $3 \times 3$  matrices with determinant equal to one and let  $\mathcal{G} = so(3)$  be the algebra of  $3 \times 3$  real skew-symmetric matrices. Consider the following basis for  $\mathcal{G}$ :

$$\mathcal{A}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \mathcal{A}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \mathcal{A}_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (21)$$

Then:

$$[\mathcal{A}_1, \mathcal{A}_2] = \mathcal{A}_3, [\mathcal{A}_1, \mathcal{A}_3] = -\mathcal{A}_2, [\mathcal{A}_2, \mathcal{A}_3] = \mathcal{A}_1. \quad (22)$$

The algebra  $\mathcal{G}$  is simple, thus, the Wei-Norman representation (2) is only local (defined when  $\cos \gamma_2 \neq 0$ ) with coefficients:

$$\begin{pmatrix} \dot{\gamma}_1 \\ \dot{\gamma}_2 \\ \dot{\gamma}_3 \end{pmatrix} = \begin{pmatrix} \sec \gamma_2 \cos \gamma_3 & -\sec \gamma_2 \sin \gamma_3 & 0 \\ \sin \gamma_3 & \cos \gamma_3 & 0 \\ -\tan \gamma_2 \cos \gamma_3 & \tan \gamma_2 \sin \gamma_3 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}. \quad (23)$$

From (2.9) and (23) we can compute  $Ad_{g^{-1}}\mathcal{A}_i$ :

$$\begin{aligned}
Ad_{g^{-1}}\mathcal{A}_1 &= \cos \gamma_2 \cos \gamma_3 \mathcal{A}_1 - \cos \gamma_2 \sin \gamma_3 \mathcal{A}_2 + \sin \gamma_2 \mathcal{A}_3, \\
Ad_{g^{-1}}\mathcal{A}_2 &= (\sin \gamma_1 \sin \gamma_2 \cos \gamma_3 + \cos \gamma_1 \sin \gamma_3) \mathcal{A}_1 \\
&\quad + (-\sin \gamma_1 \sin \gamma_2 \sin \gamma_3 + \cos \gamma_1 \cos \gamma_3) \mathcal{A}_2 - \sin \gamma_1 \cos \gamma_2 \mathcal{A}_3, \\
Ad_{g^{-1}}\mathcal{A}_3 &= (-\cos \gamma_1 \sin \gamma_2 \cos \gamma_3 + \sin \gamma_1 \sin \gamma_3) \mathcal{A}_1 \\
&\quad + (\cos \gamma_1 \sin \gamma_2 \sin \gamma_3 + \sin \gamma_1 \cos \gamma_3) \mathcal{A}_2 + \cos \gamma_1 \cos \gamma_2 \mathcal{A}_3.
\end{aligned} \quad (24)$$

Consider now the 2-module kinematic chain on  $G$ , ( $\ell = 3$ ). The system kinematics of equations (3), (4) apply.

From (22) we can see that there are three possible 2-dimensional subspace  $h$  of  $\mathcal{G}$  that can generate the whole algebra under Lie bracketing:

$$\begin{aligned} h_3 &= \text{sp}\{\mathcal{A}_1, \mathcal{A}_2\} = \text{Ker}(\mathcal{A}_3^b), \quad h_2 = \text{sp}\{\mathcal{A}_1, \mathcal{A}_3\} = \text{Ker}(\mathcal{A}_2^b), \\ \text{and } h_1 &= \text{sp}\{\mathcal{A}_2, \mathcal{A}_3\} = \text{Ker}(\mathcal{A}_1^b), \end{aligned} \quad (25)$$

We consider only  $h_3 \subset \mathcal{G}$ . The nonholonomic constraints  $\xi_i \in h_3$  can, then, be expressed as:

$$\mathcal{A}_3^b(\xi_i) = 0, \quad i = 1, 2, 3. \quad (26)$$

Equation (5) holds with  $\Xi_1$  and  $\Xi_2$  defined as above and with:

$$\begin{aligned} A_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \mathcal{A}_3^b(Ad_{(g_2^{-1}g_3)^{-1}}\mathcal{A}_1) & \mathcal{A}_3^b(Ad_{(g_2^{-1}g_3)^{-1}}\mathcal{A}_2) & \mathcal{A}_3^b(Ad_{(g_2^{-1}g_3)^{-1}}\mathcal{A}_3) & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \sin \gamma_2^{2,3} & -\sin \gamma_1^{2,3} \cos \gamma_2^{2,3} & \cos \gamma_1^{2,3} \cos \gamma_2^{2,3} & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} A_2 &= \begin{pmatrix} 0 & 0 & 1 \\ \mathcal{A}_3^b(Ad_{(g_1^{-1}g_2)^{-1}}\mathcal{A}_1) & \mathcal{A}_3^b(Ad_{(g_1^{-1}g_2)^{-1}}\mathcal{A}_2) & \mathcal{A}_3^b(Ad_{(g_1^{-1}g_2)^{-1}}\mathcal{A}_3) \\ \mathcal{A}_3^b(Ad_{(g_1^{-1}g_3)^{-1}}\mathcal{A}_1) & \mathcal{A}_3^b(Ad_{(g_1^{-1}g_3)^{-1}}\mathcal{A}_2) & \mathcal{A}_3^b(Ad_{(g_1^{-1}g_3)^{-1}}\mathcal{A}_3) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 1 \\ \sin \gamma_2^{1,2} & -\sin \gamma_1^{1,2} \cos \gamma_2^{1,2} & \cos \gamma_1^{1,2} \cos \gamma_2^{1,2} \\ \sin \gamma_2^{1,3} & -\sin \gamma_1^{1,3} \cos \gamma_2^{1,3} & \cos \gamma_1^{1,3} \cos \gamma_2^{1,3} \end{pmatrix}. \end{aligned}$$

The nonholonomic singularities of the system are the configurations where:

$$\begin{aligned} \det(A_2) &= \mathcal{A}_3^b(Ad_{(g_1^{-1}g_2)^{-1}}\mathcal{A}_1)\mathcal{A}_3^b(Ad_{(g_1^{-1}g_3)^{-1}}\mathcal{A}_2) - \mathcal{A}_3^b(Ad_{(g_1^{-1}g_3)^{-1}}\mathcal{A}_1)\mathcal{A}_3^b(Ad_{(g_1^{-1}g_2)^{-1}}\mathcal{A}_2) \\ &= -\sin \gamma_2^{1,2} \sin \gamma_1^{1,3} \cos \gamma_2^{1,3} + \sin \gamma_2^{1,3} \sin \gamma_1^{1,2} \cos \gamma_2^{1,2} = 0. \end{aligned}$$

### 3.4 Nonholonomic Kinematic Chains on the Special Linear Group $SL(2)$

Let  $G = SL(2)$  be the Special Linear group of real  $2 \times 2$  matrices with determinant one and let  $\mathcal{G} = sl(2)$  be the algebra of real  $2 \times 2$  matrices of trace zero. Consider the following basis for  $\mathcal{G}$ :

$$\mathcal{A}_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \mathcal{A}_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \mathcal{A}_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (27)$$

Then:

$$[\mathcal{A}_1, \mathcal{A}_2] = 2\mathcal{A}_3, [\mathcal{A}_1, \mathcal{A}_3] = -\mathcal{A}_1, [\mathcal{A}_2, \mathcal{A}_3] = \mathcal{A}_2. \quad (28)$$

The Wei–Norman representation (2) is only local (defined when  $e^{\gamma_3} \neq 0$ ) with coefficients:

$$\begin{pmatrix} \dot{\gamma}_1 \\ \dot{\gamma}_2 \\ \dot{\gamma}_3 \end{pmatrix} = \begin{pmatrix} e^{\gamma_3} & 0 & 0 \\ \gamma_2^2 e^{\gamma_3} & e^{-\gamma_3} & 0 \\ -2\gamma_2 e^{\gamma_3} & 0 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} \quad (29)$$

(See however comments in (Wei & Norman [1964]) and their Theorem 3. A global representation of  $SL(2)$  can be obtained using  $\{\mathcal{A}_1, \mathcal{A}_1 - \mathcal{A}_2, \mathcal{A}_3\}$  as a basis).

From (2.9) and (29) we can compute  $Ad_{g^{-1}}\mathcal{A}_i$ :

$$\begin{aligned} Ad_{g^{-1}}\mathcal{A}_1 &= e^{-\gamma_3}\mathcal{A}_1 - \gamma_2^2 e^{\gamma_3}\mathcal{A}_2 + 2\gamma_2\mathcal{A}_3, \\ Ad_{g^{-1}}\mathcal{A}_2 &= -\gamma_1^2 e^{-\gamma_3}\mathcal{A}_1 + (\gamma_1\gamma_2 + 1)^2 e^{\gamma_3}\mathcal{A}_2 - 2\gamma_1(\gamma_1\gamma_2 + 1)\mathcal{A}_3, \\ Ad_{g^{-1}}\mathcal{A}_3 &= \gamma_1 e^{-\gamma_3}\mathcal{A}_1 - \gamma_2(\gamma_1\gamma_2 + 1)e^{\gamma_3}\mathcal{A}_2 + (2\gamma_1\gamma_2 + 1)\mathcal{A}_3. \end{aligned} \quad (30)$$

Consider now the 2-module kinematic chain on  $G$ , ( $\ell = 3$ ). The system kinematics of equations (3), (4) apply.

From (28) we can see that there are two possible 2-dimensional subspaces  $h$  of  $\mathcal{G}$  that can generate the whole algebra under Lie bracketing:

$$h_3 = \text{sp}\{\mathcal{A}_1, \mathcal{A}_2\} = \text{Ker}(\mathcal{A}_3^b) \text{ and } h_{1,2} = \text{sp}\{\mathcal{A}_3, \mathcal{A}_1 + \mathcal{A}_2\} = \text{Ker}(\mathcal{A}_1^b - \mathcal{A}_2^b). \quad (31)$$

We first consider  $h_3 \subset \mathcal{G}$ . The nonholonomic constraints  $\xi_i \in h_3$  can, then, be expressed as:

$$\mathcal{A}_3^b(\xi_i) = 0, \quad i = 1, 2, 3. \quad (32)$$

Equation (5) holds with  $\Xi_1$  and  $\Xi_2$  defined as above and with:

$$\begin{aligned} A_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \mathcal{A}_3^b(Ad_{(g_2^{-1}g_3)^{-1}}\mathcal{A}_1) & \mathcal{A}_3^b(Ad_{(g_2^{-1}g_3)^{-1}}\mathcal{A}_2) & \mathcal{A}_3^b(Ad_{(g_2^{-1}g_3)^{-1}}\mathcal{A}_3) & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 2\gamma_2^{2,3} & -2\gamma_1^{2,3}(\gamma_1^{2,3}\gamma_2^{2,3} + 1) & 2\gamma_1^{2,3}\gamma_2^{2,3} + 1 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned}
A_2 &= \begin{pmatrix} 0 & 0 & 1 \\ \mathcal{A}_3^b(Ad_{(g_1^{-1}g_2)^{-1}}\mathcal{A}_1) & \mathcal{A}_3^b(Ad_{(g_1^{-1}g_2)^{-1}}\mathcal{A}_2) & \mathcal{A}_3^b(Ad_{(g_1^{-1}g_2)^{-1}}\mathcal{A}_3) \\ \mathcal{A}_3^b(Ad_{(g_1^{-1}g_3)^{-1}}\mathcal{A}_1) & \mathcal{A}_3^b(Ad_{(g_1^{-1}g_3)^{-1}}\mathcal{A}_2) & \mathcal{A}_3^b(Ad_{(g_1^{-1}g_3)^{-1}}\mathcal{A}_3) \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 1 \\ 2\gamma_2^{1,2} & -2\gamma_1^{1,2}(\gamma_1^{1,2}\gamma_2^{1,2} + 1) & 2\gamma_1^{1,2}\gamma_2^{1,2} + 1 \\ 2\gamma_2^{1,3} & -2\gamma_1^{1,3}(\gamma_1^{1,3}\gamma_2^{1,3} + 1) & 2\gamma_1^{1,3}\gamma_2^{1,3} + 1 \end{pmatrix}.
\end{aligned}$$

The nonholonomic singularities of the system are the configurations where:

$$\begin{aligned}
\det(A_2) &= \mathcal{A}_3^b(Ad_{(g_1^{-1}g_2)^{-1}}\mathcal{A}_1)\mathcal{A}_3^b(Ad_{(g_1^{-1}g_3)^{-1}}\mathcal{A}_2) - \mathcal{A}_3^b(Ad_{(g_1^{-1}g_3)^{-1}}\mathcal{A}_1)\mathcal{A}_3^b(Ad_{(g_1^{-1}g_2)^{-1}}\mathcal{A}_2) \\
&= -4\gamma_2^{1,2}\gamma_1^{1,3}(\gamma_1^{1,3}\gamma_2^{1,3} + 1) + 4\gamma_2^{1,3}\gamma_1^{1,2}(\gamma_1^{1,2}\gamma_2^{1,2} + 1) = 0.
\end{aligned}$$

Now we consider the subspace  $h_{1,2} \subset \mathcal{G}$  in (31). The nonholonomic constraints can, then, be expressed as:

$$(\mathcal{A}_1^b - \mathcal{A}_2^b)(\xi_i) = 0, \quad i = 1, 2, 3. \quad (33)$$

Equation (5) holds with  $\Xi_1$  and  $\Xi_2$  defined as above and with:

$$\begin{aligned}
A_1 &= \begin{pmatrix} 0 & 0 \\ 1 & -1 \\ (\mathcal{A}_1^b - \mathcal{A}_2^b)(Ad_{(g_2^{-1}g_3)^{-1}}\mathcal{A}_1) & (\mathcal{A}_1^b - \mathcal{A}_2^b)(Ad_{(g_2^{-1}g_3)^{-1}}\mathcal{A}_2) \\ & & 0 & 0 & 0 \\ & & 0 & 0 & 0 \\ & & (\mathcal{A}_1^b - \mathcal{A}_2^b)(Ad_{(g_2^{-1}g_3)^{-1}}\mathcal{A}_3) & 1 & -1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 1 & -1 \\ e^{-\gamma_3^{2,3}} + (\gamma_2^{2,3})^2 e^{\gamma_3^{2,3}} & -(\gamma_1^{2,3})^2 e^{-\gamma_3^{2,3}} - (\gamma_1^{2,3}\gamma_2^{2,3} + 1)^2 e^{\gamma_3^{2,3}} \\ & & 0 & 0 & 0 \\ & & 0 & 0 & 0 \\ & & \gamma_1^{2,3} e^{-\gamma_3^{2,3}} + \gamma_2^{2,3}(\gamma_1^{2,3}\gamma_2^{2,3} + 1) e^{\gamma_3^{2,3}} & 1 & -1 & 0 \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
A_2 &= \begin{pmatrix} 1 & -1 \\ (\mathcal{A}_1^b - \mathcal{A}_2^b)(Ad_{(g_1^{-1}g_2)^{-1}}\mathcal{A}_1) & (\mathcal{A}_1^b - \mathcal{A}_2^b)(Ad_{(g_1^{-1}g_2)^{-1}}\mathcal{A}_2) \\ (\mathcal{A}_1^b - \mathcal{A}_2^b)(Ad_{(g_1^{-1}g_3)^{-1}}\mathcal{A}_1) & (\mathcal{A}_1^b - \mathcal{A}_2^b)(Ad_{(g_1^{-1}g_3)^{-1}}\mathcal{A}_2) \\ & 0 \\ & (\mathcal{A}_1^b - \mathcal{A}_2^b)(Ad_{(g_1^{-1}g_2)^{-1}}\mathcal{A}_3) \\ & (\mathcal{A}_1^b - \mathcal{A}_2^b)(Ad_{(g_1^{-1}g_3)^{-1}}\mathcal{A}_3) \end{pmatrix} \\
&= \begin{pmatrix} 1 & -1 \\ e^{-\gamma_3^{1,2}} + (\gamma_2^{1,2})^2 e^{\gamma_3^{1,2}} & -(\gamma_1^{1,2})^2 e^{-\gamma_3^{1,2}} - (\gamma_1^{1,2}\gamma_2^{1,2} + 1)^2 e^{\gamma_3^{1,2}} \\ e^{-\gamma_3^{1,3}} + (\gamma_2^{1,3})^2 e^{\gamma_3^{1,3}} & -(\gamma_1^{1,3})^2 e^{-\gamma_3^{1,3}} - (\gamma_1^{1,3}\gamma_2^{1,3} + 1)^2 e^{\gamma_3^{1,3}} \\ & 0 \\ & \gamma_1^{1,2} e^{-\gamma_3^{1,2}} + \gamma_2^{1,2}(\gamma_1^{1,2}\gamma_2^{1,2} + 1)e^{\gamma_3^{1,2}} \\ & \gamma_1^{1,3} e^{-\gamma_3^{1,3}} + \gamma_2^{1,3}(\gamma_1^{1,3}\gamma_2^{1,3} + 1)e^{\gamma_3^{1,3}} \end{pmatrix}.
\end{aligned}$$

The nonholonomic singularities of the system are the configurations where:

$$\begin{aligned}
\det(A_2) &= (\mathcal{A}_1^b - \mathcal{A}_2^b)(Ad_{(g_1^{-1}g_2)^{-1}}(\mathcal{A}_1 + \mathcal{A}_2))(\mathcal{A}_1^b - \mathcal{A}_2^b)(Ad_{(g_1^{-1}g_3)^{-1}}\mathcal{A}_3) \\
&\quad - (\mathcal{A}_1^b - \mathcal{A}_2^b)(Ad_{(g_1^{-1}g_3)^{-1}}(\mathcal{A}_1 + \mathcal{A}_2))(\mathcal{A}_1^b - \mathcal{A}_2^b)(Ad_{(g_1^{-1}g_2)^{-1}}\mathcal{A}_3) \\
&= \left( e^{-\gamma_3^{1,2}} + (\gamma_2^{1,2})^2 e^{\gamma_3^{1,2}} - (\gamma_1^{1,2})^2 e^{-\gamma_3^{1,2}} - (\gamma_1^{1,2}\gamma_2^{1,2} + 1)^2 e^{\gamma_3^{1,2}} \right) \\
&\quad \cdot \left( \gamma_1^{1,3} e^{-\gamma_3^{1,3}} + \gamma_2^{1,3}(\gamma_1^{1,3}\gamma_2^{1,3} + 1)e^{\gamma_3^{1,3}} \right) \\
&\quad - \left( e^{-\gamma_3^{1,3}} + (\gamma_2^{1,3})^2 e^{\gamma_3^{1,3}} - (\gamma_1^{1,3})^2 e^{-\gamma_3^{1,3}} - (\gamma_1^{1,3}\gamma_2^{1,3} + 1)^2 e^{\gamma_3^{1,3}} \right) \\
&\quad \cdot \left( \gamma_1^{1,2} e^{-\gamma_3^{1,2}} + \gamma_2^{1,2}(\gamma_1^{1,2}\gamma_2^{1,2} + 1)e^{\gamma_3^{1,2}} \right) = 0.
\end{aligned}$$

## 4 Conclusions

In this paper we introduce the concept of *G-snakes*, which is a class of kinematic chains with nonholonomic constraints evolving on a Lie group. Shape variations of the system modules induce a snake-like global motion of the system. We provide the framework upon which motion planning strategies based on periodic shape variations can be developed. A concrete mechanical realization is associated with  $G = SE(2)$ . We offer a catalogue of low-dimensional possibilities.



Further extensions of this work include the study of geometric phase for each of the groups that we discuss, as well as the study of optimal control problems related to the optimal choice of shape variations that will achieve motion between two desired configurations.

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